1. Show that the uniform limit of a sequence of continuous functions is continuous: Let (f_n) be a seq of cts on X ($\subseteq \mathbb{R}$, or a topological space) such that $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ satisfying

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N, \; \& \; \forall x \in X.$$

Show that f is continuous at each $x_0 \in X$.

2. Let $F = \overset{N}{\underset{n=1}{\dot{\bigcup}}} F_n$, disjoint closed sets $F_1, \cdots, F_n$.
Let $f : F \to \mathbb{R}$ be such that $f|_{F_n}$ is cts, $\forall n$.
Show that f is cts.

3. Let $F_n \subseteq (n, n+1]$ be closed ($\mathbb{R} \setminus F_n$ open) $\forall n \in \mathbb{N}$, and let $F = \dot{\bigcup}_{n \in \mathbb{N}} F_n$.
Show that $f : F \to \mathbb{R}$ is continuous if each $f|_{F_n}$ is cts. (Can the condition $F_n \subseteq (n, n+1]$ be weakened to $F_n \subseteq \mathbb{R}$?)

4. Let $G = \bigcup_{n=1}^{\infty} I_n$, countable disjoint open intervals I_n, and let $F := \mathbb{R} \setminus G$. Let $x < y < z$ with $x, z \in F$ and $y \in I_n =: (a_n, b_n)$. Show that $a_n \in F$, $b_n \in F$, $x \le a_n$, and $b_n \le z$.

5. Let G, I_n, F be as in Q4, and let $f : \mathbb{R} \to \mathbb{R}$ be such that $f|_F$ and $f|_{\bar{I}_n}$ be continuous, $\forall n \in \mathbb{N}$ ($\bar{I}_n$ denotes the closure of I_n). Suppose further that the graph of $f|_{\bar{I}_n}$ is a line-segment. Show that f is continuous (By symmetry, need only show that f is right-continuity at each $x_0 \in \mathbb{R}$: $\lim_{x \to x_0+} f(x) = f(x_0)$; i.e. $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon \;\; \forall x \in (x_0, x_0 + \delta)$.

This is evident if $x_0 \in G$ (so $\exists n \in \mathbb{N}$ s.t. $x_0 \in I_n$). We may hence assume that $x_0 \in F$, and there are three cases to consider:

(a) $\exists\, \delta > 0$ s.t. $(x_0, x+\delta) \subseteq F$ (so $[x_0, x+\delta] \subseteq F$)

(b) $\exists\, \delta > 0$ s.t. $(x_0, x_0+\delta) \subseteq G$ (so $(x_0, x_0+\delta) \subseteq I_n$ for some n)

(c) $(x_0, x_0+\delta)$ intersects F and G, $\forall\, \delta > 0$.

Hint: For case (a), you use the continuity of $f|_F$.

For case (b), you use the continuity of $f|_{[x_0, x_0+\delta]}$.

For case (c), let $\varepsilon > 0$. $\exists\, \delta_0 > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ $\forall\, x \in F \cap [x_0, x_0+\delta_0]$ as $f|_F$ is continuous at x_0. By the assumption in case (c) and consider smaller $\delta_0 > 0$ if necessary, we may assume that $x_0 + \delta_0 \in F$. Show that if $x \in G \cap (x_0, x_0+\delta_0)$, then $\exists!\, n \in \mathbb{N}$ with $x \in (a_n, b_n)$. Since $x_0, x_0+\delta_0 \in F$, one has (?)

$x_0 \leq a_n < x < b_n \leq x_0 + \delta_0$ and $a_n, b_n \in F$,

$|f(\cdot) - f(x_0)| < \varepsilon$ at a_n, b_n & so at x.